EJERCICIO ELECTRODINÁMICA CUÁNTICA **CAPÍTULO 4**

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Ejercicio 4. Calcular la contribución a J^0 del campo A en la teoría QED.

El lagrangiano QED es:

$$\mathcal{L}_{QED} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - q\bar{\Psi}\gamma^\mu A_\mu \Psi - \bar{\Psi} m \Psi - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

La contribución a J^0 del campo A es:

$$J^0 = \frac{\partial \mathcal{L}_{QED}}{\partial(\partial_0 A)} \delta A - \left(\frac{\partial \mathcal{L}_{QED}}{\partial(\partial_0 A)} \partial_a A \right) \delta x^a$$

Como la transformación es una rotación:

$$\delta A^a = \omega^a{}_b A^b \qquad \delta x^a = \omega^a{}_b x^b \qquad a = 1,2,3$$

Calculamos:

$$\frac{\partial \mathcal{L}_{QED}}{\partial(\partial_0 A)} = -\tfrac{1}{4} \frac{\partial\, F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A)} \qquad \text{(solo } F_{\mu\nu}F^{\mu\nu} \text{ tiene } \partial_0 A^a\text{)}$$

$$F_{\mu\nu}F^{\mu\nu} = 2(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{13}F^{13} + F_{23}F^{23})$$

Solo nos interesan los términos que tienen 0:

$$F_{\mu\nu}F^{\mu\nu} = -2(F_{01}F_{01} + F_{02}F_{02} + F_{03}F_{03} + \ldots\ldots)$$

$$F_{\mu\nu}F^{\mu\nu} = -2[(\partial_0 A_1 - \partial_1 A_0)^2 + (\partial_0 A_2 - \partial_2 A_0)^2 + (\partial_0 A_3 - \partial_3 A_0)^2 + \ldots\ldots]$$

$$- \frac{1}{4} \frac{\partial F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A)} = - \frac{1}{4} \left(\frac{\partial F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A^1)} , \frac{\partial F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A^2)} , \frac{\partial F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A^3)} \right)$$

$$- \frac{1}{4} \frac{\partial F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A)} = (\partial_0 A_1 - \partial_1 A_0 , \partial_0 A_2 - \partial_2 A_0 , \partial_0 A_3 - \partial_3 A_0)$$

$$- \frac{1}{4} \frac{\partial F_{\mu\nu}F^{\mu\nu}}{\partial(\partial_0 A)} = (- \partial_0 A^1 - \partial_1 A^0 , - \partial_0 A^2 - \partial_2 A^0 , - \partial_0 A^3 - \partial_3 A^0) = - \partial_0 \mathbf{A} - \mathbf{\nabla} A^0 = \mathbf{E}$$

(en negrita significa trivector)

Entonces:

$$\frac{\partial \mathscr{L}_{QED}}{\partial(\partial_0 A)} = - \mathbf{\Pi} = \mathbf{E} \qquad \frac{\partial \mathscr{L}_{QED}}{\partial(\partial_0 A^a)} = \Pi_a = - \Pi^a = - E_a = E^a$$

$\mathbf{\Pi} = (\Pi^1, \Pi^2, \Pi^3)$

$\mathbf{E} = (E^1, E^2, E^3)$

Luego:

$$J^0 = \frac{\partial \mathscr{L}_{QED}}{\partial(\partial_0 A)} \delta A - \left(\frac{\partial \mathscr{L}_{QED}}{\partial(\partial_0 A)} \right) \partial_a A \, \delta x^a = \Pi_a \, \delta A^a - \Pi_d \, \partial_a A^d \, \delta x^a = - E_a \delta A^a + E_d \partial_a A^d \delta x^a$$

$$J^0 = \Pi_a \, \omega^a{}_b A^b - \Pi_d \, \delta x^a \, \partial_a \, A^d = \Pi_a \, \omega^a{}_b A^b - \Pi_d \, \omega^a{}_b x^b \, \partial_a \, A^d$$

$$J^0 = \Pi^a \, \omega_{ab} A^b + \Pi_d \, \omega_{ab} x^b \, \partial_a A^d = \Pi^a \, \varepsilon_{abc} \, \omega^c A^b - \Pi_d \, \varepsilon_{cba} \omega^c x^b \, \partial_a A^d \qquad \omega_{ab} = \varepsilon_{abc} \omega^c$$

$$J^0 = \mathbf{\Pi} \cdot (\mathbf{A} \times \boldsymbol{\omega}) - \Pi^d \, \boldsymbol{\omega} \cdot (\mathbf{r} \times \mathbf{\nabla}) \, A_d = \boldsymbol{\omega} \cdot (\mathbf{\Pi} \times \mathbf{A}) - i \boldsymbol{\omega} \cdot \Pi^d (\mathbf{r} \times [-i \mathbf{\nabla}]) A_d$$

$$J^0 = \boldsymbol{\omega} \cdot \left(\mathbf{\Pi} \times \mathbf{A} - i \Pi^d (\mathbf{r} \times \mathbf{p}) A_d \right) = \boldsymbol{\omega} \cdot \left(\mathbf{\Pi} \times \mathbf{A} - i \Pi^d \, \mathbf{L} \, A_d \right)$$

$\mathbf{L} = (L_x, L_y, L_z)$ operador

$$J^0 = \boldsymbol{\omega}\cdot\left(\boldsymbol{\Sigma}_{fotón} + \mathbf{L}_{fotón}\right)$$

$\boldsymbol{\Sigma}_{fotón} = \boldsymbol{\Pi} \times \mathbf{A} = - \mathbf{E} \times \mathbf{A}$ (densidad momento angular espín del fotón)

$\mathbf{L}_{fotón} = - i\Pi^d \, \mathbf{L} \, A_d = iE^d \, \mathbf{L} \, A_d$ (densidad momento angular orbital del fotón)